$(\varphi, \psi) = \langle \varphi | \psi \rangle$

basis: $\{e_i\}$ $i = 1, \dots, n$

$\langle e_i | e_j \rangle = \delta_{ij}$

$\Rightarrow |\psi\rangle = \sum_i \alpha_i |e_i\rangle$

$\langle e_k | \psi \rangle = \langle e_k | \sum_i \alpha_i | e_i \rangle = \sum_i \alpha_i \langle e_k | e_i \rangle$

$= \sum_i \alpha_i \delta_{ki}$

$= \alpha_k$

$|\psi\rangle = \underbrace{\sum |e_i\rangle\langle e_i|}_{\text{id matrix} = I} \psi\rangle$

$\langle \varphi | \psi \rangle \longrightarrow \boxed{\langle \varphi | \psi \rangle}$

ψ is ket

φ is bra

$|\psi\rangle \to |\varphi\rangle$

$\varphi^* = \langle \varphi |$

mapping $\langle \varphi | : \mathcal{H} \to \mathbb{C}$

$f(|\psi\rangle) = \langle \varphi | \psi \rangle$

bra of φ, ψ

$\langle \varphi | \psi \rangle$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$\langle a | b \rangle = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n$

$\langle a | = (a_1^*, a_2^*, \dots, a_n^*)$

$|b\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Dagger operation

$|\psi\rangle = \alpha_1 |\varphi_1\rangle + \alpha_2 |\varphi_2\rangle \to \langle \psi | = \alpha_1^* \langle \varphi_1 | + \alpha_2^* \langle \varphi_2 |$

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operators

basis: $\{|i\rangle\}$

$|\varphi\rangle = \sum_i |i\rangle a_i$

$|\psi\rangle = \sum_i |i\rangle b_i$

operator $\hat{\Omega} \Rightarrow$ one-to-one mapping a variable $b \longrightarrow \boxed{\sum_i \hat{\Omega} |i\rangle a_i = \sum_i |i\rangle b_i}$

$\hat{\Omega} |i\rangle = |j\rangle$

repeat from left with $\langle k |$: $\sum_i \langle k | \hat{\Omega} | i \rangle a_i = \sum_i \langle k | i \rangle b_i = b_k$

define matrix elements: $\Omega_{ki} = \langle k | \hat{\Omega} | i \rangle$

claim: $\hat{\Omega} = \sum_k |k\rangle \Omega_{kl} \langle l |$

verify $\langle k | \hat{\Omega} | i \rangle = \sum_{kl} \Omega_{kl} \langle k | l \rangle \langle l | i \rangle = \sum_{kl} \Omega_{kl} \delta_{kl} \delta_{li}$

$= \Omega_{ki}$

$|\uparrow\rangle \to |\uparrow\downarrow\rangle \to \psi^*$

$|\downarrow\rangle \to \psi$

$|\text{spin up}\rangle$

all/all this

Spin $\frac{1}{2}$ $\quad \uparrow, \uparrow \to |\uparrow\rangle \times |\uparrow\rangle$

$\frac{\hbar}{2}$ $\frac{\hbar}{2}$ $\frac{\hbar}{2}$ $\frac{\hbar}{2}$

$\hbar$: different spin 1/2 particles

$\hbar$: two identical spin 1/2 particles (Pauli exclusion principle)

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state $|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle|\psi_2\rangle$ (cross product) $\longleftrightarrow$ with/without effect $\longleftrightarrow$ state $|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle|\psi_2\rangle$

wave function of particle $\psi(\vec{r}_1, \vec{r}_2, t)$

time dependent Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

$H = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t)$

$|\psi(\vec{r}_1, \vec{r}_2, t)|^2 d^3r_1 d^3r_2$ = prob to find particle 1 in volume element d^3r_1 and " " 2 in " " d^3r_2

$-\frac{\hbar^2}{2m}\nabla_1^2\psi - \frac{\hbar^2}{2m}\nabla_2^2\psi + \dots$

separate rel coords of mass coordinate

separate SE easily

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

$$\boxed{\begin{aligned} \psi_R &= e^{i\vec{k}\vec{R}} \\ E_R &= \frac{\hbar^2 k^2}{2M} \end{aligned}}$$

$$\boxed{-\frac{\hbar^2}{2\mu}\nabla^2\psi_r + V(\vec{r})\psi_r = E_r\psi_r}$$

$E = E_R + E_r$

$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)$

only valid for distinguishable particles

two particles are indistinguishable

$\psi_s(\vec{r}_1, \vec{r}_2) = A\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) + \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right)$

bosons: integer spin, photons, mesons — sym. fn.

fermions: half-integer spin, electrons, protons — antisym. fn.

$A\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right) = 0$

2 identical fermions cannot be in the same state (Pauli exclusion principle)

Swapping operator P

$P f(\vec{r}_1, \vec{r}_2) = f(\vec{r}_2, \vec{r}_1)$ — swapping coordinates

$P^2 f(\vec{r}_1, \vec{r}_2) = f(\vec{r}_1, \vec{r}_2)$ — P^2 the eigenvalue 1

$[\vec{P}, H] = 0 \Longrightarrow$ solutions of Schrödinger equation symmetric or antisymmetric

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antisymmetric — spin antisymmetric

$S^2 |s m\rangle = \hbar^2 s(s+1) |s m\rangle$ — eq. 4.135

$S_z |s m\rangle = \hbar m |s m\rangle$

$S_\pm |s m\rangle = \hbar \sqrt{s(s+1) - m(m\pm1)} \; |s (m\pm1)\rangle$

$S_x = \frac{1}{2}(S_+ + S_-)$

$S_y = \frac{1}{2i}(S_+ - S_-)$

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2 non-interacting particles, mass m, in 1d well

$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2) \qquad E_{n_1 n_2} = (n_1^2 + n_2^2)K$

Ground state $\psi_{11}(x_1, x_2) = \frac{1}{2}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$

$E_{11} = E_1 = 5K$