Lecture 3

Tuesday, 8 September 2020 08:47

$|s\ m_s\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$ spin up ↑

$|s\ m_s\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$ spin down ↓

$a\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + b\,|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$ write as $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

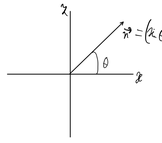
$\chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$\chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$\chi = a\chi_+ + b\chi_-$

S_x

$$\chi = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\chi_+^{(x)} + \frac{1}{\sqrt{2}}\chi_-^{(x)} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$\vec{n} = (\sin\theta, \cos\theta)$

$$S_\theta = \vec{n}\cdot\vec{S} = (\cos\theta, \sin\theta)\cdot(S_x, S_z)$$

$$= \frac{1}{2}\hbar\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we want to find the eigenvalues of S_θ

eigenvalue equation

simplified notation

$$\frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2\sqrt{2}}\bar{\lambda}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{vmatrix} 1-\bar{\lambda} & 1 \\ 1 & -1-\bar{\lambda} \end{vmatrix} = 0$$

$$(-\bar{\lambda}+1)(-\bar{\lambda}-1) - 1\cdot 1 = 0$$

$$\bar{\lambda}^2 - 1 - 1 = 0$$

$$\bar{\lambda}^2 = 2$$

$$\bar{\lambda} = \pm\sqrt{2}$$

$$\lambda = \frac{\hbar}{2\sqrt{2}}(\pm\sqrt{2}) = \pm\frac{\hbar}{2} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2\sqrt{2}}\begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

take +, the

$$\frac{1}{\sqrt{2}}\alpha + \frac{1}{\sqrt{2}}\beta = \alpha \longrightarrow \left(1-\frac{1}{\sqrt{2}}\right)\alpha = \frac{1}{\sqrt{2}}\beta \longrightarrow \beta = (\sqrt{2}-1)\alpha$$

$$\frac{1}{\sqrt{2}}\alpha - \frac{1}{\sqrt{2}}\beta = \beta$$

$$\frac{1}{\sqrt{2}}\alpha - \alpha + \frac{1}{\sqrt{2}}\alpha = \alpha\sqrt{2} - \alpha$$

$$\frac{2}{\sqrt{2}}\alpha = \alpha\sqrt{2}$$

$$\frac{1}{\sqrt{2}}(\alpha+\beta) = \alpha$$

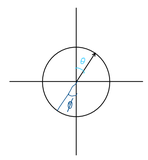
$$\beta = \alpha(\sqrt{2}-1) \longrightarrow \chi_+^{(\theta)} = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

$$|\uparrow\rangle = \chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_+^{(\theta)}\chi_+^{(\theta)} + c_-^{(\theta)}\chi_-^{(\theta)}$$

inner product

$$c_+^{\theta} = (1\ \ 0)\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}\frac{1}{\sqrt{4-2\sqrt{2}}} \Longrightarrow p_+^{(\theta)} = |c_+^{\theta}|^2 = \frac{1}{4-2\sqrt{2}} \simeq 0.854$$

Generally $\chi_+^{(\theta)} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$; $\chi_-^{(\theta)} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$



$$|+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$$

H-atom: 2 spin $\frac{1}{2}$ particles (electron + proton

no orbital angular momentum

4 states: $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ $\quad |s_1\ m_1\ s_2\ m_2\rangle$

$s_1, s_2 = \frac{1}{2}$ $\quad m_1, m_2 = \pm\frac{1}{2}$

$S^{(1)}$ only acts on $\chi^{(1)}$

$S^{(2)}$ only acts on $\chi^{(2)}$

$$S_z\chi^{(1)}\chi^{(2)} = (S_z^{(1)} + S_z^{(2)})\chi^{(1)}\chi^{(2)} = (S_z^{(1)}\chi^{(1)})\chi^{(2)} + \chi^{(1)}(S_z^{(2)}\chi^{(2)})$$

$$= (\hbar m_1\chi^{(1)})\chi^{(2)} + \chi^{(1)}(\hbar m_2\chi^{(2)}) = \hbar(m_1+m_2)\chi^{(1)}\chi^{(2)}$$

$$S_- = S_-^{(1)} + S_-^{(2)}, \quad S_z = S_z^{(1)} + S_z^{(2)}$$

possible total spins: $s = (s_1+s_2), (s_1+s_2-1), \ldots |s_1-s_2|$

total number of states $\sum_{s=|s_1-s_2|}^{(s_1+s_2)}(2s+1)$