Lecture 2

Friday, February 4, 2022 11:08

Hilbert space = vector space where some functions 'live'

vector addition $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$

vector scalar multiplication $a|\alpha\rangle = |\gamma\rangle$

$$|\alpha\rangle \to \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

inner product $\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \dots + a_N^* b_N$

linear transformation

$$|\beta\rangle = T|\alpha\rangle \to \underline{b} = \underline{T}\,\underline{a} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{21} & T_{22} & \dots & T_{2N} \\ \vdots & & & \vdots \\ T_{N1} & T_{N2} & \dots & T_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

$$\boxed{x\, g_y(x) = y\, g_y(x)} \Rightarrow g_y(x) = A\,\delta(x-y)$$

eigenvalue

$$\int_{-\infty}^{\infty} g_{y'}(x) g_y(x)\,dx = |A|^2 \int_{-\infty}^{\infty} \delta(x-y')\delta(x-y)\,dx = |A|^2 \delta(y-y')$$

choose $A=1 \Rightarrow g_y(x) = \delta(x-y)$

$$\langle g_{y'} | g_y \rangle = \delta(y-y')$$

$$f(x) = \int_{-\infty}^{\infty} c(y) g_y(x)\,dy = \int_{-\infty}^{\infty} c(y)\delta(x-y)\,dy = c(x)$$

$$\langle g_{y'} | f \rangle = \int c(y) \langle g_{y'} | g_y \rangle dy = \int c(y)\delta(y'-y)\,dy = c(y') = f(y')$$

coefficients corresponds to the function itself

If a measurement on a dynamical var, we will always find a eigenvalue of the eigenfunction

for discrete spectra $|c_i|^2$, with $c_i = \langle f_i | \psi \rangle$

for continuous spectra $|c(y)|^2 dy$, with $c(y) = \langle f_y | \psi \rangle$

all these mathematical configurations

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle [\hat{A}, \hat{B}] \rangle\right)^2$$

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad L_x = yp_z - zp_y$$

$$L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad L_y = zp_x - xp_z$$

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \qquad L_z = xp_y - yp_x$$

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

$$L_\pm = L_x \pm iL_y$$

$$L_\pm \Rightarrow |c| = \sqrt{\frac{1}{2}}\,\hbar$$

Eigenfunctions lead to the $Y_{lm}(\theta, \phi)$

L^2 and L_z commute

$\vdots$

algebraically they lead to the half-integer values

algebraically they lead to only the integer

$$e^{i2\pi(l+1)} = e^{i2\pi l} \text{ only for integer}$$

for spin, it follows that half-integer values are possible

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classical:

$\vec{L} = \vec{r} \times \vec{p}$ orbital angular momentum because of orbit of electrons of mass

$\vec{S} = r\omega$ angular momentum because of the rotational motion of mass

i.e. QM: spin is intrinsic property of mass!

classical: $p = IA = \frac{q}{2\pi/\omega}\pi r^2 = \frac{q}{2m} m r^2 = \frac{q}{2m} L$

(QM) magnetic moment

$$\mu = \gamma \vec{S} = \gamma \frac{q}{2m} S$$

$$\gamma = 2.00232\,\gamma_{QM} \frac{q}{2m}$$

$$\frac{\mu_e}{S_e} = 1 + \frac{\alpha^2}{12} + c_1\left(\frac{\alpha}{\pi}\right)^2 + c_2\left(\frac{\alpha}{\pi}\right)^3 + \dots$$

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036} \Rightarrow \gamma = 2.00232\,\gamma_{QM}$$

quantum electrodynamics

$$H = -\vec{\mu} \cdot \vec{B}$$

$$F_z = \frac{\partial}{\partial z}(\vec{\mu} \cdot \vec{B})_z = \frac{\partial \mu_z}{\partial z}$$

$$\frac{\partial B_z}{\partial z} < 0$$

$\mu_z < 0 \Rightarrow S_z > 0 \to$ deflection up $\to$ spin up

$\mu_z > 0 \Rightarrow S_z < 0 \to$ deflection down $\to$ spin down

only separable

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

$$\vec{S}^2 |s\,m_s\rangle = \hbar^2 s(s+1)|m_s\rangle$$

$$S_z |s\,m_s\rangle = \hbar m_s |s\,m_s\rangle$$

$$S_\pm |s\,m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}\,|s\,m_s \pm 1\rangle$$

$$S_\pm = S_x \pm iS_y$$

[illegible]

recognized that $e^{i2\pi(l+1)} = e^{i2\pi l}$

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$m_s = -s, -s+1, \dots, s-1, s$

electron, proton $s = \frac{1}{2}$

photon $s = 1$

graviton $s = 2$

[illegible] (higgs, gluons) $s = \frac{1}{2}$

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separable two-state i.e. S_z (measurement)

spin $\frac{1}{2}$ $\quad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ right spin up?

$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ right down?

spin $\hat{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$X_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1+\rangle \qquad X_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1-\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli spin matrices

$\vec{S}^2, S_x, S_y, S_z$ are hermitian (eigenvalues are real, observable)

S_+, S_- are not hermitian

eigenspace S_z

$X_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvalue $+\frac{\hbar}{2}$

$X_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenvalue $-\frac{\hbar}{2}$

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measurement $\begin{pmatrix} a \\ b \end{pmatrix}$

probability finding $+\frac{\hbar}{2} = |a|^2$

probability finding $-\frac{\hbar}{2} = |b|^2$

$$|a|^2 + |b|^2 = 1$$

Larmor precession

magnetic field in z-direction $\vec{B} = B_0\hat{z}$

$$H = \gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

eigenvalue same as that of S_z: X_+ will eigenvalue $E_+ = -\frac{\gamma B_0 \hbar}{2}$

X_- will eigenvalue $E_- = +\frac{\gamma B_0 \hbar}{2}$

initial state

$$\chi(t) = a X_+ e^{-iE_+ t/\hbar} + b X_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{-i\gamma B_0 t/2} \\ b e^{+i\gamma B_0 t/2} \end{pmatrix}, \quad \chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$