9.2a

$$\psi(x) = e^{if(x)/\hbar}$$

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi$$

$$\frac{d^2}{dx^2} e^{if(x)/\hbar} = -\frac{p^2}{\hbar^2} e^{if(x)/\hbar}$$

$$\frac{d}{dx}\left(\frac{i}{\hbar}\frac{df(x)}{dx} e^{if(x)/\hbar}\right) = -\frac{p^2}{\hbar^2} e^{if(x)/\hbar}$$

$$\frac{i}{\hbar}\frac{d^2f(x)}{dx^2} e^{if(x)/\hbar} + \frac{i^2}{\hbar^2}\left(\frac{df(x)}{dx}\right)^2 e^{if(x)/\hbar} = -\frac{p^2}{\hbar^2} e^{if(x)/\hbar}$$

$$\frac{i}{\hbar} f'' - \frac{1}{\hbar^2}(f')^2 = -\frac{p^2}{\hbar^2}$$

$$i\hbar f'' - (f')^2 + p^2 = 0$$

9.2b

$$f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x)$$

$$f'(x) = f_0'(x) + \hbar f_1'(x) + \hbar^2 f_2'(x)$$

$$f''(x) = f_0''(x) + \hbar f_1''(x) + \hbar^2 f_2''(x)$$

$$(f'(x))^2 = (f_0'(x))^2 + \hbar^2(f_1'(x))^2 + \hbar^4(f_2'(x))^2 + 2\hbar f_0'(x)f_1'(x) + 2\hbar^3 f_1'(x)f_2'(x) + 2\hbar^2 f_0'(x)f_2'(x)$$

filling in (and leaving out the x-dependence and terms of order higher than 2) gives

9.2 b
part 2

$$i\hbar\left(f_0'' + \hbar f_1''\right) \qquad - f_0'^2 - \hbar^2 f_1'^2 - 2\hbar f_0' f_1' - 2\hbar^2 f_0' f_2' + p^2 = 0$$

$\hbar^0$: $p^2 = \left(f_0'\right)^2$

$\hbar^1$: $i f_0'' - 2 f_0' f_1' = 0 \rightarrow i f_0'' = 2 f_0' f_1'$

$\hbar^2$: $i f_1'' - \left(f_1'\right)^2 - 2 f_0' f_2' = 0 \rightarrow i f_1'' = 2 f_0' f_2' + \left(f_1'\right)^2$

9.2 C

from the $\hbar^0$-equation, we get

$$p = \pm f_0'$$

$$\text{so } f_0 = \pm \int p(x)\,dx + \text{constant}$$

from the $\hbar^1$-equation, we get

$$\frac{1}{p}\frac{dp}{dx} = \frac{d}{dx}\ln(p)$$

$$f_1' = \frac{i}{2}\frac{f_0''}{f_0'} = \frac{i}{2}\frac{\pm p'}{\pm p} = \frac{i}{2}\frac{d}{dx}\ln(p)$$

$$f_1 = \frac{i}{2}\ln p + \text{constant}$$

$$\Psi = e^{if/\hbar} = \exp\left(\frac{i}{\hbar}\left(\pm\int p(x)\,dx + \hbar\frac{i}{2}\ln(p) + \text{constant}\right)\right)$$

we got f_1 from the $\hbar^1$ equation, meaning (as f is a power series in $\hbar$), that we need to multiply by $\hbar$.

$$= \exp\left(\pm\frac{i}{\hbar}\int p\,dx\right) p^{-\frac{1}{2}} e^{\frac{i}{\hbar}K}$$

$$= \frac{C}{\sqrt{p}}\exp\left(\pm\frac{i}{\hbar}\int p\,dx\right)$$

This is the WKB approximated wavefunction. $\square$