Exercise 8.1

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$\psi(x) = Ae^{-bx^2}$

$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2bx^2} dx = |A|^2\, 2\int_0^{\infty} e^{-2bx^2} dx = 2|A|^2 \sqrt{\pi} \left(\frac{\frac{1}{\sqrt{2b}}}{2}\right) = |A|^2 \sqrt{\frac{\pi}{2b}}$

$|A|^2 = \sqrt{\frac{2b}{\pi}}$

$A = \sqrt[4]{\frac{2b}{\pi}}$

$-2bxe^{-bx^2} \rightarrow -2be^{-bx^2} + 2bx \cdot 2bxe^{-bx^2}$ ($4b^2x^2$)

$\langle T \rangle = -\frac{\hbar^2}{2m} |A|^2 \int_{-\infty}^{\infty} e^{-bx^2} \frac{d^2}{dx^2}\left(e^{-bx^2}\right) dx = -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\, 2 \cdot 4b^2 \int_0^{\infty} x^2 e^{-2bx^2} dx = -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(8b^2 \cdot \sqrt{\pi}\, \frac{2!}{1!}\left(\frac{\frac{1}{\sqrt{2b}}}{2}\right)^3 - 2b\sqrt{\frac{\pi}{2b}}\right)$

$-2b\int_0^{\infty} e^{-2bx^2} = -4b\sqrt{\frac{\pi}{2b}}$

$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(8b^2\sqrt{\pi} \cdot 2\frac{1}{16b\sqrt{2b}} - 2b\sqrt{\frac{\pi}{2b}}\right)$

$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(\sqrt{\pi}\sqrt{b}\frac{1}{\sqrt{2}} - 2\sqrt{\pi}\sqrt{b}\frac{1}{\sqrt{2}}\right)$

$= -\frac{\hbar^2}{2m} b(1-2) = \frac{\hbar^2 b}{2m}$ ✓

$\langle V \rangle = \langle \psi | V(x) | \psi \rangle = |A|^2 \int_{-\infty}^{\infty} e^{-bx^2} \alpha |x|\, e^{-bx^2} dx = \alpha |A|^2 \int_{-\infty}^{\infty} |x| e^{-2bx^2} dx$

$= \alpha |A|^2 \left(\int_0^{\infty} x e^{-2bx^2} dx + \int_{-\infty}^{0} -x\, e^{-2bx^2} dx\right)$

$= \alpha |A|^2\, 2\int_0^{\infty} x e^{-2bx^2} dx$

$= 2\alpha |A|^2 \frac{1}{2}\left(\frac{1}{\sqrt{2b}}\right)^2 = \frac{\alpha |A|^2}{2b} = \frac{\alpha}{2b}\sqrt{\frac{2b}{\pi}} = \frac{\alpha}{\sqrt{2b\pi}}$

$\langle H \rangle = \langle T \rangle + \langle V \rangle = \frac{\hbar^2 b}{2m} + \frac{\alpha}{\sqrt{2\pi b}}$

$\frac{\partial \langle H \rangle}{\partial b} = \frac{\hbar^2}{2m} + \frac{\alpha}{\sqrt{2\pi}}\, b^{-\frac{3}{2}} \cdot -\frac{1}{2} = 0$

$\frac{\alpha}{2\sqrt{2\pi}}\, b^{-\frac{3}{2}} = \frac{\hbar^2}{2m}$

$b^{-\frac{3}{2}} = \frac{\hbar^2\, 2\sqrt{2\pi}}{2m\alpha} = \frac{\hbar^2\sqrt{2\pi}}{m\alpha}$

$b = \left(\frac{m\alpha}{\hbar^2\sqrt{2\pi}}\right)^{\frac{2}{3}}$

$\langle H \rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{m\alpha}{\hbar^2\sqrt{2\pi}}\right)^{\frac{2}{3}} + \frac{\alpha}{\sqrt{2\pi}}\left(\frac{m\alpha}{\hbar^2\sqrt{2\pi}}\right)^{\frac{2}{3}\cdot -\frac{1}{2}}$

$= \frac{\hbar^2}{2m}\left(\frac{m\alpha}{\hbar^2\sqrt{2\pi}}\right)^{\frac{2}{3}} + \frac{\alpha}{\sqrt{2\pi}}\left(\frac{\hbar^2\sqrt{2\pi}}{m\alpha}\right)^{\frac{1}{3}}$

$= \frac{\hbar^2}{2m}\frac{m^{\frac{2}{3}}\alpha^{\frac{2}{3}}}{\hbar^{\frac{4}{3}}\sqrt[3]{2\pi}} + \frac{\alpha}{\sqrt{2\pi}}\frac{\hbar^{\frac{2}{3}}(2\pi)^{\frac{1}{6}}}{m^{\frac{1}{3}}\alpha^{\frac{1}{3}}}$

$= \left(1 + \frac{1}{2}\right)\left(\frac{\alpha^2\hbar^2}{m\, 2\pi}\right)^{\frac{1}{3}}$

$= \frac{3}{2}\left(\frac{\alpha^2\hbar^2}{2\pi m}\right)^{\frac{1}{3}}$

$V = \alpha x^4$

$\langle V \rangle = \langle \psi | V(x) | \psi \rangle$

$= \alpha |A|^2 \int_{-\infty}^{\infty} x^4 e^{-2bx^2} dx$

$= 2\alpha |A|^2 \sqrt{\pi}\; 12 \left(\frac{1}{2\sqrt{2b}}\right)^5$ (from $-\infty$ to ∞; $12 = \frac{4!}{2!}$)

$= 24\alpha \sqrt{\frac{2b}{\pi}}\sqrt{\pi}\,\frac{1}{2^5}\frac{1}{2^2b^2}\frac{1}{\sqrt{2b}}$

$= \frac{24}{2^5}\alpha\frac{1}{2^2b^2}$

$= \frac{24}{128}\frac{\alpha}{b^2}$

$= \frac{3}{16}\frac{\alpha}{b^2}$

$\langle H \rangle = \langle T \rangle + \langle V \rangle$

$= \frac{\hbar^2 b}{2m} + \frac{3}{16}\frac{\alpha}{b^2}$

$\frac{\partial \langle H \rangle}{\partial b} = \frac{\hbar^2}{2m} + \frac{3}{16}\alpha \cdot -2\, b^{-3} = 0$

$\frac{\hbar^2}{2m} = \frac{3}{8}\alpha\frac{1}{b^3}$

$b^3 = \frac{3}{4}\frac{m\alpha}{\hbar^2}$

$\langle H \rangle_{min} = \frac{\hbar^2}{2m}\left(\frac{3\alpha m}{4\hbar^2}\right)^{\frac{1}{3}} + \frac{3}{16}\alpha\left(\frac{4\hbar^2}{3\alpha m}\right)^{\frac{2}{3}}$

$= \frac{\hbar^2}{2m}\frac{3^{\frac{1}{3}}\alpha^{\frac{1}{3}}m^{\frac{1}{3}}}{4^{\frac{1}{3}}\hbar^{\frac{2}{3}}} + \frac{3}{16}\alpha\frac{4^{\frac{2}{3}}\hbar^{\frac{4}{3}}}{3^{\frac{2}{3}}\alpha^{\frac{2}{3}}m^{\frac{2}{3}}}$

$= \left(\frac{3^{\frac{1}{3}}}{2\cdot 4^{\frac{1}{3}}} + \frac{3}{16}\frac{4^{\frac{2}{3}}}{3^{\frac{2}{3}}}\right)\left(\frac{\alpha\hbar^4}{m^2}\right)^{\frac{1}{3}}$

$= \left(\frac{1}{2} + \frac{3}{16}\cdot\frac{4}{3}\right)\left(\frac{3}{4}\frac{\alpha\hbar^4}{m^2}\right)^{\frac{1}{3}}$

$= \left(\frac{1}{2} + \frac{12}{48}\right)\left(\frac{3}{4}\frac{\alpha\hbar^4}{m^2}\right)^{\frac{1}{3}}$

$= \left(\frac{1}{2} + \frac{1}{4}\right)\left(\frac{3}{4}\frac{\alpha\hbar^4}{m^2}\right)^{\frac{1}{3}}$

$= \frac{3}{4}\left(\frac{3}{4}\frac{\alpha\hbar^4}{m^2}\right)^{\frac{1}{3}}$