$\boxed{7.5a}$

$$E_n^2 = \sum_{m\neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$H' = \alpha\,\delta(x - a/2)$$

$$\psi_n^0 = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$|\langle \psi_m^0 | H' | \psi_n^0 \rangle| = \left| \frac{2}{a} \int_{-\infty}^{\infty} \alpha\,\delta(x - a/2) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx \right|$$

$$= \left| \frac{2}{a}\,\alpha \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \right|$$

$$= \begin{cases} 0 & \text{if } n \text{ is even or } m \text{ is even} \\ \pm \frac{2}{a}\alpha & \text{if } m \text{ and } n \text{ are both odd} \end{cases}$$

↳ squaring drops the minus sign

$$E_n^2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ \left(\frac{2\alpha}{a}\right)^2 \frac{2ma^2}{\pi^2\hbar^2} \sum_{m\neq n,\,odd} \frac{1}{n^2 - m^2} & \text{if } n \text{ is odd} \end{cases}$$

E_n^0

$$\frac{1}{n^2 - m^2} = \frac{1}{2n}\left(\frac{1}{m+n} - \frac{1}{m-n}\right) = \frac{1}{2n} \frac{m-n - m-n}{m^2 - n^2} = \frac{-2n}{m^2 - n^2}\,\frac{1}{2n}$$

for $n = 1$

$$\text{sum} = \frac{1}{2} \sum_{3,5,7\ldots} \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} \ldots - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} \ldots\right) = \frac{1}{2} \cdot -\frac{1}{2} = -\frac{1}{4}$$

for $n = 3$

$$\text{sum} = \frac{1}{6} \sum_{1,5,7} = \frac{1}{6}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} \ldots - \left(-\frac{1}{2}\right) - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} \ldots\right) = \frac{1}{6} \cdot -\frac{1}{6} = -\frac{1}{36}$$

generally: $\text{sum} = -\frac{1}{(2n)^2}$

Then $$E_n = \begin{cases} 0 & \text{for } n \text{ even} \\ -2m\left(\frac{\alpha}{\pi\hbar n}\right)^2 & \text{for } n \text{ odd} \end{cases}$$

7.5b

$V(x) = \frac{1}{2} k x^2$ $\qquad H' = \frac{1}{2}\varepsilon k x^2$

$E_n = \left(\frac{1}{2}+n\right)\hbar\omega = \left(\frac{1}{2}+n\right)\hbar\sqrt{\frac{k}{m}} \rightarrow \left(\frac{1}{2}+n\right)\hbar\sqrt{\frac{(1+\varepsilon)k}{m}}$

$$E_n^2 = \sum_{m\neq n} \frac{|\langle\psi_n^0|H'|\psi_m^0\rangle|^2}{E_n^0 - E_m^0}$$

$$\langle\psi_m^0|H'|\psi_n^0\rangle \equiv \frac{1}{2}k\varepsilon \ \langle m|x^2|n\rangle$$

$$\langle m|x^2|n\rangle = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 \langle m|(\hat{a}_+ + \hat{a}_-)^2|n\rangle$$

$$= \frac{\hbar}{2m\omega} \langle m|\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2|n\rangle$$

$$= \frac{\hbar}{2m\omega}\Big(\langle m|n+2\rangle\sqrt{n+1}\sqrt{n+2} + \sqrt{n}\sqrt{n}\langle m|n\rangle + \sqrt{n+1}\sqrt{n+1}\langle m|n\rangle + \sqrt{n}\sqrt{n-1}\langle m|n-2\rangle\Big)$$

$$\overset{\text{if } m\neq n}{=} \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right)$$

$$E_n^2 = \left(\frac{\varepsilon\hbar\omega}{4}\right)^2 \sum_{m\neq n} \frac{\left(\sqrt{(n+1)(n+2)}\,\delta_{m,n+2} + \sqrt{n(n-1)}\,\delta_{m,n-2}\right)^2}{(n+\frac{1}{2})\hbar\omega - (m+\frac{1}{2})\hbar\omega}$$

cross terms drop out due to unequal δ-functions

$$\Rightarrow \frac{\varepsilon^2\hbar\omega}{16} \sum_{m\neq n} \frac{(n+1)(n+2)\delta_{m,n+2} + n(n-1)\delta_{m,n-2}}{n-m}$$

7.5 b
part 2

$$E_n^2 = \frac{\varepsilon^2 \hbar\omega}{16}\left(\frac{(n+1)(n+2)}{n-n-2} + \frac{n(n-1)}{n-n+2}\right)$$

$$= \frac{\varepsilon^2 \hbar\omega}{16}\left(\frac{(n+1)(n+2)}{-2} + \frac{n(n-1)}{2}\right)$$

$$= \frac{\varepsilon^2 \hbar\omega}{32}\left(n^2 - n - n^2 - n - 2n - 2\right)$$

$$= \frac{\varepsilon^2 \hbar\omega}{32}\left(-4n-2\right) = -\frac{\varepsilon^2 \hbar\omega}{8}\left(n+\frac{1}{2}\right)$$

, which agrees with the ε^2 term in the exact solution