$\boxed{7.12a}$

$\varepsilon = 0$ gives

$V_0 - E = 0$ ✓ ~~$(V_0 - E)(2V_0 - E) - V_0^2\varepsilon^2 = 0$~~

$E = V_0$ ✓ $\quad V_0 - E = 0$ ✓ $2V_0 - E = 0$

$V_0 = E$ ✓ $E = 2V_0$

$E_1 = V_0 \,;\, \chi_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$E_2 = V_0 \,;\, \chi_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$E_3 = 2V_0 \,;\, \chi_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{7.12b}$

$V_0 - \varepsilon - \lambda = 0 \rightarrow \lambda = V_0 - \varepsilon$

$(V_0 - \lambda)(2V_0 - \lambda) - V_0^2\varepsilon^2 = 0$

$\lambda^2 - 3V_0\lambda + 2V_0^2 - V_0^2\varepsilon^2 = 0$

$D = V_0^2 9 - 4V_0^2(2 - \varepsilon^2)$

$\lambda = \frac{V_0 3 \pm V_0\sqrt{9 - 4(2-\varepsilon^2)}}{2} = \frac{3 \pm \sqrt{1+4\varepsilon^2}}{2} V_0$

$\lambda_2 = \frac{V_0}{2}\left(3 - \sqrt{1+4\varepsilon^2}\right)$ | $\lambda_3 = \frac{V_0}{2}\left(3 + \sqrt{1+4\varepsilon^2}\right)$

(assuming $\varepsilon \ll 1$ expand)

$\lambda_2 \approx \frac{V_0}{2}\left(3 - (1 + 2\varepsilon^2)\right)$ | $\lambda_3 \approx V_0(2 + \varepsilon^2)$

$= V_0(1 - \varepsilon^2)$

$\partial_\varepsilon = \frac{1}{2} \cdot \frac{8\varepsilon}{\sqrt{1+4\varepsilon^2}} = \frac{4\varepsilon}{\sqrt{\dots}}$

$\partial_{\varepsilon 2} = \frac{4}{\sqrt{\dots}} - \frac{1}{2}\frac{8\varepsilon}{(\dots)^{\frac{3}{2}}} \cdot 4\varepsilon$

7.12c

nondegenerate eigenvalue: $E_3 = 2V_0$ with eigenvector $\chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$E_3^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} V_0\varepsilon & 0 & 0 \\ 0 & 0 & V_0\varepsilon \\ 0 & V_0\varepsilon & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ V_0\varepsilon \\ 0 \end{pmatrix} = 0$$

$$E_3^2 = \sum_{\substack{m=1,2 \\ (n=3) \\ m \neq 3}} \frac{|\langle \chi_m | H' | \chi_3 \rangle|^2}{E_3^0 - E_m^0}$$

$$\langle \chi_1 | H' | \chi_3 \rangle = \varepsilon V_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \varepsilon V_0 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

$$= 0$$

$$\langle \chi_2 | H' | \chi_3 \rangle = \varepsilon V_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \varepsilon V_0 \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \varepsilon V_0$$

$$E_3^2 = \frac{|\varepsilon V_0|^2}{2V_0 - V_0} = \frac{\varepsilon^2 V_0^2}{V_0} = \varepsilon^2 V_0$$

up to second order, then $E_3 = E_3^0 + E_3^1 + E_3^2 = 2V_0 + \varepsilon^2 V_0 = V_0(2+\varepsilon^2)$

which is the same as in b (for χ_3)

7.12d

We have two-fold degeneracy, so we can use the strategy described in § 7.2.1

$$W_{ij} = \langle \psi_i^0 | H' | \psi_j^0 \rangle$$

let a=1, b=2, then $W_{aa} = \langle \chi_1 | H' | \chi_1 \rangle = V_0\varepsilon \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$= V_0\varepsilon \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = V_0\varepsilon$$

$(W_{ba} =)\ W_{ab} = \langle \chi_1 | H' | \chi_2 \rangle = V_0\varepsilon \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$= V_0\varepsilon \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$W_{bb} = \langle \chi_2 | H' | \chi_2 \rangle = V_0\varepsilon \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

So, $W = \begin{pmatrix} -V_0\varepsilon & 0 \\ 0 & 0 \end{pmatrix}$

eigenvalues of W are perturbed energies: (first-order correction)

$$(-V_0\varepsilon - E) \cdot -E = 0$$

so the corrections are $E_2^1 = 0 \lor E_1^1 = -V_0\varepsilon$

and up to first order, we get $E_1 = V_0 - V_0\varepsilon$, $E_2 = V_0$, which is the same as in b