Exercise A.31

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$$\exp\begin{pmatrix}0&1&3\\0&0&4\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&1&3\\0&0&4\\0&0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0&4\\0&0&0\\0&0&0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&1&5\\0&1&4\\0&0&1\end{pmatrix}$$

$$\exp\begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-\theta^2&0\\0&-\theta^2\end{pmatrix} + \frac{1}{6}\begin{pmatrix}0&-\theta^3\\\theta^3&0\end{pmatrix} + \frac{1}{24}\begin{pmatrix}\theta^4&0\\0&\theta^4\end{pmatrix}\cdots$$

$$= \begin{pmatrix}1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4\cdots & \theta-\frac{1}{6}\theta^3\cdots\\ -\theta+\frac{1}{6}\theta^3\cdots & 1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4\cdots\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

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if M is diagonalizable, then there exist an S, such that $SMS^{-1} = D = \begin{pmatrix}d_1&&&\emptyset\\&d_2&&\\&&d_3&\\\emptyset&&&\ddots\ d_n\end{pmatrix}$

$$Se^{M}S^{-1} = S\left(1+M+\frac{1}{2}M^2+\frac{1}{6}M^3+\dots\right)S^{-1}$$

$SS^{-1} = I$

$$= I + SMS^{-1} + \frac{1}{2}SMS^{-1}SMS^{-1} + \frac{1}{6}SMS^{-1}SMS^{-1}SMS^{-1}\dots$$

$$= I + D + \frac{1}{2}D^2 + \frac{1}{6}D^3\dots = e^{D}$$

$\det(S)\det(S^{-1}) = 1$

$$\det(e^{D}) = \det(Se^{M}S^{-1}) = \det(S)\det(e^{M})\det(S^{-1}) = \det(e^{M})$$

$$D^k = \begin{pmatrix}d_1^k&&&\\&d_2^k&&\\&&d_3^k&\\&&&\ddots\ d_n^k\end{pmatrix}$$

$$e^{D} = I + \begin{pmatrix}d_1&&&\\&d_2&&\\&&\ddots&\\&&&d_n\end{pmatrix} + \frac{1}{2}\begin{pmatrix}d_1^2&&&\\&d_2^2&&\\&&\ddots&\\&&&d_n^2\end{pmatrix} + \frac{1}{6}\begin{pmatrix}d_1^3&&&\\&d_2^3&&\\&&\ddots&\\&&&d_n^3\end{pmatrix} + \dots = \begin{pmatrix}e^{d_1}&&&\\&e^{d_2}&&\\&&\ddots&\\&&&e^{d_n}\end{pmatrix}$$

$$\det(e^{D}) = e^{\left(\sum_i^n d_i\right)} = e^{\mathrm{Tr}(D)} = e^{\mathrm{Tr}(M)}$$

Commuting matrices obey the same rules as ordinary numbers

if $S(\lambda) = e^{\lambda M}e^{\lambda N}$, then $\frac{dS}{d\lambda} = Me^{\lambda M}e^{\lambda N} + e^{\lambda M}Ne^{\lambda N} = (M+N)e^{\lambda M}e^{\lambda N} = (M+N)S$

solve DE: $\frac{dS}{d\lambda} = (M+N)S$

$S(\lambda) = Ae^{(M+N)\lambda}$, for some A

$S(0) = I \Rightarrow A = 1$

Thus $S(\lambda) = e^{(M+N)\lambda}$

$$S(1) = e^{(M+N)} = e^{M}e^{N}$$

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counterexample for non-commuting matrices

$M = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ $N = \begin{pmatrix}0&0\\-1&0\end{pmatrix}$

$M^2 = N^2 = 0$

$$[M,N] = MN - NM = \begin{pmatrix}-1&0\\0&0\end{pmatrix} - \begin{pmatrix}0&0\\0&-1\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix} \neq 0$$

$$e^{M} = I + M = \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

$$e^{N} = I + N = \begin{pmatrix}1&0\\-1&1\end{pmatrix}$$

$$e^{M}e^{N} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix} = \begin{pmatrix}0&1\\-1&1\end{pmatrix}$$

$\neq$

$$M+N = \begin{pmatrix}0&1\\-1&0\end{pmatrix} \qquad e^{M+N} = \begin{pmatrix}\cos 1 & \sin 1\\ -\sin 1 & \cos 1\end{pmatrix}$$

$$e^{iH} = \sum_{n=0}^{\infty}\frac{1}{n!}i^n H^n$$

Hermitian

$$\left(e^{iH}\right)^{\dagger} = \sum_{n=0}^{\infty}\frac{1}{n!}(-i)^n (H)^n = \sum_{n=0}^{\infty}\frac{1}{n!}(-i^{\dagger})(H)^n = e^{-iH}$$

$$\left(e^{iH}\right)^{\dagger}\left(e^{iH}\right) = e^{-iH}e^{iH} = e^{i(H-H)} = 1,$$ so e^{iH} is unitary