Exercise A.30

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a) $\langle \hat{U}\alpha | \hat{U}\beta \rangle = \langle \hat{U}^\dagger \hat{U}\alpha | \beta \rangle = \langle \alpha | \beta \rangle$

b) eigenvalue equation $\hat{U}|\alpha\rangle = \lambda|\alpha\rangle \Rightarrow \langle \hat{U}\alpha | \hat{U}\alpha \rangle = \langle \hat{U}^\dagger \hat{U}\alpha | \alpha \rangle = 1\langle \alpha|\alpha\rangle = \langle \lambda^*\alpha | \lambda\alpha\rangle = |\lambda|^2 \langle\alpha|\alpha\rangle = |\lambda|^2$

Which implies $|\lambda|^2 = 1$, or: the modulus of the eigenvalues must always be 1 for a unitary transformation

c) Suppose $\hat{U}|\alpha\rangle = \lambda|\alpha\rangle$

and $\hat{u}|\beta\rangle = u|\beta\rangle$

Then, $\langle\beta|\hat{U}|\alpha\rangle = \lambda\langle\beta|\alpha\rangle$

but also $\langle u^\dagger\beta|\alpha\rangle = \frac{1}{u}\langle\beta|\alpha\rangle$

$(\langle u^\dagger \beta|)^\dagger = \hat{u}^\dagger|\beta\rangle$

$$\boxed{\begin{aligned} |\beta\rangle &= \underbrace{\hat{U}^\dagger\hat{U}}_{=1}|\beta\rangle = \hat{u}^\dagger u|\beta\rangle \\ u^\dagger|\beta\rangle &= \tfrac{1}{u}|\beta\rangle \end{aligned}}$$

We have $|\lambda|^2 = |u|^2 = 1$

thus $|\lambda|^2 \hat{U}^\dagger|\beta\rangle = u^*|\beta\rangle$

$u^*|\beta\rangle = U^\dagger|\beta\rangle$

$\langle\hat{U}^\dagger\beta| = \hat{U}\langle\beta|$

Then $\hat{u}^\dagger|\beta\rangle = u^*|\beta\rangle$ ~~implies that $\langle\hat{u}^\dagger\beta| = \langle\beta|$~~

But then $\langle\beta|\hat{U}\alpha\rangle = \lambda\langle\beta|\alpha\rangle = \langle\hat{U}^\dagger\beta|\alpha\rangle = u\langle\beta|\alpha\rangle$

$u\langle\beta|\alpha\rangle = \lambda\langle\beta|\alpha\rangle$

$(u-\lambda)\langle\beta|\alpha\rangle = 0$

$u - \lambda = 0 \lor \langle\beta|\alpha\rangle = 0$

if $u \neq \lambda$, this implies $\langle\beta|\alpha\rangle = 0 \Rightarrow \beta$ and α are orthogonal