Exercise A.28

dinsdag 8 september 2020 22:53

$$T = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

$$T^\dagger = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

$T = T^\dagger \rightarrow T$ is hermitian

$$\begin{pmatrix} 1-\lambda & 1-i \\ 1+i & 0-\lambda \end{pmatrix}$$

$$-(1-\lambda)\lambda - (1+i)(1-i) = 0$$

$$\lambda^2 - \lambda - (1-i^2) = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda-2)(\lambda+1) = 0$$

$$\lambda = 2 \quad \vee \quad \lambda = -1$$

eigenvalues are real

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{cases} 2\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ -\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \end{cases} \rightarrow \begin{pmatrix} \alpha + \beta(1-i) \\ \alpha(1+i) \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \alpha + \beta - i\beta \\ \alpha + i\alpha \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 2\beta \end{pmatrix}$$

$$\beta(1-i) = \alpha$$

$$\beta = \frac{\alpha(1+i)}{2}$$

$$\alpha + \beta - i\beta = \alpha + \beta(1-i)$$
$$= \alpha\left(1 + \tfrac{1}{2}(1-i)(1+i)\right)$$
$$= \alpha\left(1 + \tfrac{1}{2}\cdot 2\right)$$
$$= 2\alpha$$

$$|\alpha|^2 + |\beta|^2 = 1$$
$$|(1-i)\beta|^2 + |\beta|^2 = 1$$
$$((1-i)(1+i)+1)\,|\beta|^2 = 1$$
$$3\,|\beta|^2 = 1$$
$$\beta = \frac{1}{\sqrt{3}}$$

$$a^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$\beta = -\alpha(1+i)$$

$$2\alpha = -\beta(1-i)$$
$$\alpha = -\tfrac{1}{2}\beta(1-i)$$

$$|\alpha|^2 + |-\alpha(1+i)|^2 = 1$$
$$|\alpha|^2 + (1+i)(1-i)|\alpha|^2 = 1$$
$$\alpha = \frac{1}{\sqrt{3}}$$

$$\frac{1}{4}\cdot 2|\beta|^2 + |\beta|^2 = 1$$
$$\beta = \frac{1}{\sqrt{\frac{3}{2}}} = \sqrt{\frac{2}{3}}$$

$$a^{(2)} = \frac{1}{\sqrt{6}}\begin{pmatrix} -(1-i) \\ 2 \end{pmatrix}$$

d) equation (A.21): $(S^{-1})_{ij} = (a^{(j)})_i$

$$S^{-1} = \begin{pmatrix} \frac{1-i}{\sqrt{3}} & \frac{i-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i & \frac{i-1}{\sqrt{2}} \\ 1 & \sqrt{2} \end{pmatrix}$$

$$S = \frac{1}{\det(S^{-1})}\tilde{C}$$

$$= \frac{1}{\frac{1-i}{\sqrt{3}}\frac{2}{\sqrt{6}} - \frac{i-1}{\sqrt{6}}\frac{1}{\sqrt{3}}}\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$$

$$= \frac{1}{\frac{1-i}{\sqrt{18}}}\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$$

$$=$$

for distinct eigenvalues with orthonormal eigenvectors; S is unitary and we get and T is hermitian

$$S = (S^{-1})^\dagger$$

$$S = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1+i & 1 \\ \frac{1+i}{\sqrt{2}} & \sqrt{2} \end{pmatrix}$$

$$S\,T\,S^{-1} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} \frac{1-i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1+i & 1+i \\ 1 & \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} \frac{1-i}{\sqrt{3}} + \frac{1-i}{\sqrt{3}} = \frac{2-2i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} + \frac{2-2i}{\sqrt{6}} = \frac{3-3i}{\sqrt{6}} \\ \frac{(1-i)(1+i)}{\sqrt{3}} = \frac{2}{\sqrt{3}} & \frac{(1-i)(1+i)}{\sqrt{6}} = \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+i}{\sqrt{3}}\frac{2-2i}{\sqrt{3}} + \frac{2(1-i)(1+i)}{\sqrt{18}} = \frac{4}{\sqrt{6}} + \frac{2+2i}{\sqrt{18}} & \frac{1+i}{\sqrt{3}}\,3\,\frac{1-i}{\sqrt{6}} + 2\,\frac{1+i}{6} \end{pmatrix}$$