Exercise 4.37

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a)

$S_-|1\ 0\rangle$

$S_- = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$$S_-|1\ 0\rangle = S_-\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(\big(\underbrace{S_-^{(1)}|\uparrow\rangle}_{=\hbar|\downarrow\rangle}\big)|\downarrow\rangle + |\uparrow\rangle\big(S_-^{(2)}|\downarrow\rangle\big) + \big(S_-^{(1)}|\downarrow\rangle\big)|\uparrow\rangle + |\downarrow\rangle\big(S_-^{(2)}|\uparrow\rangle\big)\Big)$$

$$= \frac{1}{\sqrt{2}}\left(\hbar|\downarrow\downarrow\rangle + 0 + 0 + \hbar|\downarrow\downarrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(2\hbar|\downarrow\downarrow\rangle\right)$$

$$= \sqrt{2}\,\hbar|\downarrow\downarrow\rangle$$

$$= \sqrt{2}\,\hbar|1\ -1\rangle \quad \square$$

b)

$$S_\pm|0\ 0\rangle = S_\pm\left(\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\big(S_\pm^{(1)}|\uparrow\rangle\big)|\downarrow\rangle + |\uparrow\rangle\big(S_\pm^{(2)}|\downarrow\rangle\big) - \big(S_\pm^{(1)}|\downarrow\rangle\big)|\uparrow\rangle - |\downarrow\rangle\big(S_\pm^{(2)}|\uparrow\rangle\big)\right)$$

$$= \frac{1}{\sqrt{2}}\begin{cases}\big(S_+^{(1)}|\uparrow\rangle\big)|\downarrow\rangle + |\uparrow\rangle\big(S_+^{(2)}|\downarrow\rangle\big) - \big(S_+^{(1)}|\downarrow\rangle\big)|\uparrow\rangle - |\downarrow\rangle\big(S_+^{(2)}|\uparrow\rangle\big)\\ \big(S_-^{(1)}|\uparrow\rangle\big)|\downarrow\rangle + |\uparrow\rangle\big(S_-^{(2)}|\downarrow\rangle\big) - \big(S_-^{(1)}|\downarrow\rangle\big)|\uparrow\rangle - |\downarrow\rangle\big(S_-^{(2)}|\uparrow\rangle\big)\end{cases} = \begin{cases}(0)|\downarrow\rangle + |\uparrow\rangle(\hbar|\uparrow\rangle) - (\hbar|\uparrow\rangle)|\uparrow\rangle - |\downarrow\rangle(0)\\ (\hbar|\downarrow\rangle)|\downarrow\rangle + 0 - 0 - |\downarrow\rangle(\hbar|\downarrow\rangle)\end{cases} = \begin{cases}0\\0\end{cases} = 0$$

c)

$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$

$$S^2|1\ 1\rangle = \left(\left(S^{(1)}\right)^2 + \left(S^{(2)}\right)^2 + 2S^{(1)}\cdot S^{(2)}\right)|\uparrow\uparrow\rangle$$

$$= \left(S^{(1)}\right)^2|\uparrow\uparrow\rangle + \left(S^{(2)}\right)^2|\uparrow\uparrow\rangle + 2S^{(1)}\cdot S^{(2)}|\uparrow\uparrow\rangle$$

$S_x\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$$= \left(\left(S^{(1)}\right)^2|\uparrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(\left(S^{(2)}\right)^2|\uparrow\rangle\right) + 2\left[(S_x|\uparrow\rangle)_1(S_x|\uparrow\rangle)_2 + (S_y|\uparrow\rangle)_1(S_y|\uparrow\rangle)_2 + (S_z|\uparrow\rangle)_1(S_z|\uparrow\rangle)_2\right]$$

$\begin{pmatrix}0\\1\end{pmatrix}$ $\quad \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\quad \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix} =$

$\frac{3}{4}\hbar^2 I =$

$$= \frac{3}{4}\hbar^2|\uparrow\uparrow\rangle + \frac{3}{4}\hbar^2|\uparrow\uparrow\rangle + 2\left(\frac{\hbar^2}{4}|\downarrow\downarrow\rangle + \frac{\hbar^2}{4}i^2|\downarrow\downarrow\rangle + \frac{\hbar^2}{4}|\uparrow\uparrow\rangle\right)$$

$\begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle$

$$= \frac{6}{4}\hbar^2|\uparrow\uparrow\rangle + \frac{2\hbar^2}{4}|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$

$$= 1(1+1)\hbar^2|1\ 1\rangle$$

$$S^2|1\ -1\rangle = \left(\left(S^{(1)}\right)^2|\downarrow\rangle\right)|\downarrow\rangle + |\downarrow\rangle\left(\left(S^{(2)}\right)^2|\downarrow\rangle\right) + 2S^{(1)}\cdot S^{(2)}|\downarrow\downarrow\rangle$$

$$= \frac{6}{4}\hbar^2|\downarrow\downarrow\rangle + 2\left(\big(S_x^{(1)}|\downarrow\rangle\big)\big(S_x^{(2)}|\downarrow\rangle\big) + \big(S_y^{(1)}|\downarrow\rangle\big)\big(S_y^{(2)}|\downarrow\rangle\big) + \big(S_z^{(1)}|\downarrow\rangle\big)\big(S_z^{(2)}|\downarrow\rangle\big)\right)$$

$$= \frac{6}{4}\hbar^2|\downarrow\downarrow\rangle + 2\left(\frac{\hbar^2}{4}|\uparrow\uparrow\rangle - \frac{\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{\hbar^2}{4}|\downarrow\downarrow\rangle\right)$$

$$= 2\hbar^2|\downarrow\downarrow\rangle$$

$$= 1(1+1)\hbar^2|1\ -1\rangle$$