Exercise 4.61

Tuesday, 8 September 2020 12:00

$$\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

(s, m_s)

$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S_\pm \left|s\ m\right\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\ \left|s\ m\pm1\right\rangle$$

$S_+\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = 0$; $S_-\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle = \sqrt{3}\hbar\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle$

$S_+\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{3}\hbar\left|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle$; $S_-\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = 2\hbar\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle$ ← $\sqrt{\tfrac{3}{2}\left(\tfrac{5}{2}\right) - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)} = \sqrt{4}$

$S_+\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = 2\hbar\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle$; $S_-\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle = \sqrt{3}\hbar\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle$ ← $\sqrt{\tfrac{3}{2}\left(\tfrac{5}{2}\right) - \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)} = \sqrt{\tfrac{12}{4}}$

$S_+\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = \sqrt{3}\hbar\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle$; $S_-\left|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle = 0$

$$S_+ = \hbar\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad S_- = \hbar\begin{pmatrix}0 & 0 & 0 & 0\\ \sqrt{3} & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}$$

$$S_x = \tfrac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}$$

$$|S_x - \lambda I| = 0$$

$$\begin{pmatrix}-\lambda & \frac{\hbar}{2}\sqrt{3} & 0 & 0\\ \frac{\hbar}{2}\sqrt{3} & -\lambda & \frac{\hbar}{2}2 & 0\\ 0 & \frac{\hbar}{2}2 & -\lambda & \frac{\hbar}{2}\sqrt{3}\\ 0 & 0 & \frac{\hbar}{2}\sqrt{3} & -\lambda\end{pmatrix}$$

~~$-\lambda\left(-\lambda\left(-\lambda\cdot-\lambda - \frac{\hbar}{2}\sqrt{3}\cdot\frac{\hbar}{2}\sqrt{3}\right) - \hbar\left(\hbar\cdot-\lambda - 0\cdot\frac{\hbar}{2}\sqrt{3}\right)\right) - \frac{\hbar}{2}\sqrt{3}\left(\frac{\hbar}{2}\sqrt{3}\left(-\lambda\cdot-\lambda - \frac{\hbar}{2}\sqrt{3}\cdot\frac{\hbar}{2}\sqrt{3}\right) - \hbar\cdot 0\right) = 0$~~

~~$-\lambda\left(-\lambda\left(\lambda^2 - \frac{\hbar^2 3}{4}\right) + \hbar^2\lambda\right) - \frac{\hbar}{2}\sqrt{3}\left(\frac{\hbar^2}{2}\sqrt{3}\left(\lambda^2 - \frac{\hbar^2 3}{4}\right)\right) = 0$~~

~~$\lambda^4 - \frac{3\lambda^2\hbar^2}{4} - \hbar^2\lambda^2 - \frac{3\hbar^3}{4}\lambda^2 - \frac{9\hbar^4}{16} = 0$~~

~~take $u = \lambda^2$~~

~~$u^2 - u\left(\hbar^2 + \frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^3\right) - \frac{9}{16}\hbar^4 = 0$~~

~~$u^2 - u\left(\frac{7}{4}\hbar^2 + \frac{3}{4}\hbar^3\right) - \frac{9}{16}\hbar^4 = 0$~~

$$\begin{vmatrix}-\lambda & \sqrt{3} & 0 & 0\\ \sqrt{3} & -\lambda & 2 & 0\\ 0 & 2 & -\lambda & \sqrt{3}\\ 0 & 0 & \sqrt{3} & -\lambda\end{vmatrix} = -\lambda\begin{vmatrix}-\lambda & 2 & 0\\ 2 & -\lambda & \sqrt{3}\\ 0 & \sqrt{3} & -\lambda\end{vmatrix} - \sqrt{3}\begin{vmatrix}\sqrt{3} & 2 & 0\\ 0 & -\lambda & \sqrt{3}\\ 0 & \sqrt{3} & -\lambda\end{vmatrix}$$

$$= -\lambda\left(-\lambda\left(\lambda^2 - 3\right) - 2\left(-2\lambda\right)\right) - \sqrt{3}\sqrt{3}\left(\lambda^2 - 3\right)$$

$$= -\lambda\left(3\lambda - \lambda^3 + 4\lambda\right) - \sqrt{3}\sqrt{3}\lambda^2 + 3\sqrt{3}\sqrt{3}$$

$$= \lambda^4 - 7\lambda^2 - 3\lambda^2 + 9$$

$$= \lambda^4 - 10\lambda^2 + 9$$

$\{u = \lambda^2\}$

$$= u^2 - 10u + 9$$

$$= (u - 9)(u - 1)$$

$$= (\lambda^2 - 9)(\lambda^2 - 1)$$

$\Downarrow$

$\lambda^2 = 9 \ \lor\ \lambda^2 = 1$

$\lambda = \pm 3 \ \lor\ \lambda = \pm 1$

$\Downarrow$

So, the eigenvalues of S_x are $\frac{3}{2}\hbar,\ \frac{1}{2}\hbar,\ -\frac{1}{2}\hbar,\ -\frac{3}{2}\hbar$