Exercise 4.32

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a) $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{vmatrix} = 0$$

$$\left(\lambda^2 - \left(\frac{\hbar}{2}\right)^2\right) = 0$$

$$\lambda^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\lambda = \frac{\hbar}{2} \vee \lambda = -\frac{\hbar}{2}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$-i\beta = \pm\alpha$$

$$\alpha = \pm i\beta$$

$$\alpha = \frac{1}{\sqrt{2}} \quad \beta = \pm\frac{i}{\sqrt{2}}$$

$$S_y\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow (S_y - \lambda I)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$|S_y - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 = \lambda^2 - \frac{\hbar^2}{4}$$

$$\lambda = \pm\frac{\hbar}{2}$$

Now put into eigenvalue equation $\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$-i\beta = \pm\alpha$$

Condition $|\alpha|^2 + |\beta|^2 = 1$ → $\alpha = \frac{1}{\sqrt{2}}$

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

for $\lambda = +\frac{\hbar}{2}$, we get $\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ for $\lambda = -\frac{\hbar}{2}$, we get $\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

b) $S_y = +\frac{\hbar}{2}$, with $c_+ = \left(\chi_+^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}(1 \;\; -i)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a - ib)$

So probability $\frac{1}{2}|a - ib|^2$

↑ c_+ absolute squared

$S_y = -\frac{\hbar}{2}$, with $c_- = \left(\chi_-^{(y)}\right)^\dagger \chi = \frac{1}{\sqrt{2}}(1 \;\; i)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a + ib)$

So probability $\frac{1}{2}|a + ib|^2$

$$\frac{1}{2}|a - ib|^2 + \frac{1}{2}|a + ib|^2 =$$

($|a-ib|^2$ ↳ $(a-ib)^*(a-ib)$)

$$\frac{1}{2}\left(|a|^2 - ia^*b - iab^* + |b|^2 + |a|^2 + ia^*b + iab^* + |b|^2\right) = |a|^2 + |b|^2 = 1$$

c) Measurement of S_y^2 will yield $\frac{\hbar^2}{4}$ with probability 1.