Exercise 4.30

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a)

$$\chi = A\begin{pmatrix}3i\\4\end{pmatrix}$$

$$|A3i|^2 + |A4|^2 = 1$$

$$9A^2 + 16A^2 = 1$$

$$25A^2 = 1$$

$$A^2 = \frac{1}{25}$$

$$A = \pm\frac{1}{5}$$

choose $A = \frac{1}{5}$

b)

transpose & complex conjugate

$$\langle S_x\rangle = \chi^\dagger S_x \chi = \frac{1}{5}\begin{pmatrix}-3i & 4\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}4\\3i\end{pmatrix}$$

$$= \frac{\hbar}{50}(-12i + 12i) = 0$$

$$\langle S_y\rangle = \chi^\dagger S_y \chi = \frac{1}{5}\begin{pmatrix}-3i & 4\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}-4i\\-3\end{pmatrix}$$

$$= \frac{\hbar}{50}(-12 - 12) = -\frac{24}{50}\hbar$$

$$= -\frac{12}{25}\hbar$$

$$\langle S_z\rangle = \chi^\dagger S_z \chi = \frac{1}{5}\begin{pmatrix}-3i & 4\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar}{50}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\-4\end{pmatrix}$$

$$= \frac{\hbar}{50}(-9i^2 - 16)$$

$$= -\frac{7\hbar}{50}$$

$$\langle S_x^2\rangle = \chi^\dagger S_x S_x \chi = \frac{1}{5}\begin{pmatrix}-3i & 4\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}4\\3i\end{pmatrix}$$

$$= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}(-9i^2 + 16)$$

$$= \frac{\hbar^2}{100}(9 + 16)$$

$$= \frac{\hbar^2 25}{100} = \frac{\hbar^2}{4}$$

$$\langle S_y^2\rangle = \chi^\dagger S_y^2 \chi = \frac{1}{5}\begin{pmatrix}-3i & 4\end{pmatrix}\left(\frac{\hbar}{2}\right)^2\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}-4i\\-3\end{pmatrix}$$

$$= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}(9 + 16) = \frac{\hbar^2}{4}$$

$$\langle S_z^2\rangle = \chi^\dagger S_z^2 \chi = \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}3i\\-4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= \frac{\hbar^2}{100}(9 + 16) = \frac{\hbar^2}{4}$$

$$\sigma_{S_x} = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$$

$$\sigma_{S_y} = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \left(-\frac{12}{25}\hbar\right)^2} = \sqrt{\frac{\hbar^2}{4} - \frac{144}{625}\hbar^2}$$

$$= \sqrt{\frac{\hbar^2 625 - 576}{2500}}$$

$$= \sqrt{\frac{49}{25\cdot 100}}\hbar^2$$

$$= \frac{7}{50}\hbar$$

$$\sigma_{S_z} = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{49}{2500}\hbar^2}$$

$$= \sqrt{\frac{625 - 49}{2500}\hbar^2}$$

$$= \frac{\hbar}{50}\sqrt{576}$$

$$= \frac{24}{50}\hbar$$

$$= \frac{12}{25}\hbar$$

d)

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7}{50}\hbar = \frac{7\hbar^2}{100} \geq \frac{\hbar}{2}|\langle S_z\rangle| = \frac{7}{2\cdot 50}\hbar^2 = \frac{7}{100}\hbar^2 \checkmark$$

$$\sigma_{S_y}\sigma_{S_z} = \frac{7\cdot 12}{50\cdot 25}\hbar^2 \geq \frac{\hbar}{2}|\langle S_x\rangle| = 0 \checkmark$$

$$\sigma_{S_z}\sigma_{S_x} = \frac{\hbar}{2}\frac{12}{25}\hbar = \frac{12}{50}\hbar^2 \geq \frac{\hbar}{2}|\langle S_y\rangle| = \frac{\hbar\, 12}{2\cdot 25}\hbar = \frac{12}{50}\hbar^2 \checkmark$$