Exercise 4.21

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commutator identities: $[\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$

$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

assuming L_x and L_y are hermitian

will be conjugated on left side of inner product

$$\langle f | L_\pm g\rangle = \langle f | L_x g\rangle \pm i \langle f | L_y g\rangle = \langle L_x f | g\rangle \pm i \langle L_y f | g\rangle = \langle L_x f | g\rangle \mp \langle i L_y f | g\rangle$$

$$= \langle (L_x \mp i L_y) f | g\rangle$$

$$= \langle L_\mp f | g\rangle$$

implying L_+ and L_- are each other's hermitian conjugate

Then, we could say that

$$\langle L_\pm f_\ell^m | L_\pm f_\ell^m\rangle = \langle f_\ell^m | L_\mp L_\pm f_\ell^m\rangle$$

$$\langle A_\ell^m f_\ell^{m+1} | A_\ell^m f_\ell^{m+1}\rangle = \langle f_\ell^m | (L^2 - L_z^2 \mp \hbar L_z) f_\ell^m\rangle$$

$$|A_\ell^m|^2 \langle f_\ell^{m+1} | f_\ell^{m+1}\rangle = \langle f_\ell^m | \underbrace{(\hbar^2 \ell(\ell+1) - (\hbar m)^2 \mp \hbar^2 m)}_{\text{a constant}} f_\ell^m\rangle$$

$$|A_\ell^m|^2 = (\hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m) \underbrace{\langle f_\ell^m | f_\ell^m\rangle}_{=1}$$

$$|A_\ell^m|^2 = \hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m$$

$$A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m \pm 1)}$$

replacing A with B and choosing the other part of $\pm$ gives:

$$A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}$$

$$B_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m-1)}$$