Exercise 2.11

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} = \alpha\, e^{-\xi^2}$$

factor $\frac{1}{2}$ disappears because we take ψ_0^2

$$\frac{d\xi}{dx} = \sqrt{\frac{m\omega}{\hbar}} \Rightarrow dx = \sqrt{\frac{\hbar}{m\omega}}\, d\xi$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 |\psi_0(x)|^2 dx = \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\xi^2} dx = \alpha^2 \left(\sqrt{\frac{\hbar}{m\omega}}\right)^3 \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi = \alpha^2 \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}} \sqrt{\pi}\, 2 \left(\frac{1}{2}\right)^3$$

we expect from $-\infty < x < \infty$ and of course

$$= \sqrt{\frac{\pi}{2}} \sqrt{\left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}}} \quad \frac{2^{\frac{3}{2}}}{2^3}$$

$$= \frac{\hbar}{2\omega}$$

?

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi_0 \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi_0\, dx$$

$$\frac{d\xi}{dx} = \sqrt{\frac{m\omega}{\hbar}} \Rightarrow dx = \sqrt{\frac{\hbar}{m\omega}}\, d\xi$$

$$= -\hbar^2\alpha^2 \int_{-\infty}^{\infty} e^{-\xi^2} \frac{\partial^2}{\partial x^2} e^{-\frac{\xi^2}{2}} dx$$

$$= -\hbar^2\alpha^2 \int_{-\infty}^{\infty} e^{-\xi^2} \frac{\partial^2}{\partial \xi^2} e^{-\frac{\xi^2}{2}} d\xi \quad \times \sqrt{\frac{m\omega}{\hbar}}$$

$$\frac{\partial}{\partial \xi} e^{-\frac{\xi^2}{2}} = -\xi e^{-\frac{\xi^2}{2}}$$

$$\frac{\partial^2}{\partial \xi^2} e^{-\frac{\xi^2}{2}} = -e^{-\frac{\xi^2}{2}} + \xi^2 e^{-\frac{\xi^2}{2}}$$

we know because $\hbar$ is represented by ξ (because ξ is equivalent to $\hbar$ in the dimensionless units $\frac{m\omega}{\hbar}$ and $\frac{\hbar}{m\omega}$), this accordingly, thus $\frac{m\omega}{\hbar}$

$$= -\hbar^2\alpha^2 \sqrt{\frac{m\omega}{\hbar}} \int_{-\infty}^{\infty} \left(\xi^2 - 1\right) e^{-\xi^2} = \frac{-m\omega\hbar}{2} \left(\frac{\sqrt{\pi}}{2} - \sqrt{\pi}\right) = \frac{m\omega\hbar}{2}$$

$$= -\hbar^2 \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{m\omega}{\hbar}}$$

$$= \frac{-m\omega\hbar}{\sqrt{\pi}}$$

$n = 1 \Longrightarrow$

$$\psi_1 = \frac{1}{\sqrt{1!}} \hat{a}_+ \psi_0$$

$$= \left(\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\right)\left(\alpha\, e^{-\frac{\xi^2}{2}}\right)$$

$$= \sqrt{\frac{m\omega}{\hbar}}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\left(\alpha\, e^{-\frac{\xi^2}{2}} \cdot -\xi \cdot \frac{d\xi}{dx}\right) + m\omega x\, \alpha\, e^{-\frac{\xi^2}{2}}\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \cdot -\frac{m\omega}{\hbar}x + m\omega x \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\xi^2}{2}}\right) 2$$

$$= \sqrt{\frac{2m\omega}{\hbar}}\, \alpha\, e^{-\frac{\xi^2}{2}}$$

$$= \sqrt{2}\,\alpha\, \xi\, e^{-\frac{\xi^2}{2}}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$d\xi = \sqrt{\frac{m\omega}{\hbar}}\, dx$$

$\frac{1}{2}$ for symmetry dx to $d\xi$ and $\frac{1}{2}$ for symmetry x^2 to ξ^2 from ex. around 0

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 |\psi_1|^2 dx = 2\alpha^2 \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi = 2\alpha^2 \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}} \sqrt{\pi}\, \frac{24}{2} \left(\frac{1}{2}\right)^5$$

standard integral, $n = 2$

$$= \alpha^2 \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}} \frac{48}{2^5} \sqrt{\pi}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}} \frac{48}{32} \sqrt{\pi}$$

$$= \frac{\hbar}{m\omega}\, \frac{3}{2}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \sqrt{2}\,\alpha\, \xi\, e^{-\frac{\xi^2}{2}} \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \sqrt{2}\,\alpha\, \xi\, e^{-\frac{\xi^2}{2}} dx$$

$$= -2\alpha^2 \hbar^2 \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} \xi\, e^{-\frac{\xi^2}{2}} \left(\frac{\partial^2}{\partial x^2}\left(\xi\, e^{-\frac{\xi^2}{2}}\right)\right) d\xi$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \sqrt{\frac{m\omega}{\hbar}} \quad \frac{\partial \xi}{\partial x}$$

$$\frac{\partial}{\partial \xi}\left(\xi\, e^{-\frac{\xi^2}{2}}\right) = e^{-\frac{\xi^2}{2}} - \xi^2 e^{-\frac{\xi^2}{2}}$$

$$\frac{\partial^2}{\partial \xi^2}\left(\xi\, e^{-\frac{\xi^2}{2}}\right) = -\xi e^{-\frac{\xi^2}{2}} - 2\xi e^{-\frac{\xi^2}{2}} + \xi^3 e^{-\frac{\xi^2}{2}}$$

$$= -2\alpha^2 \hbar^2 \sqrt{\frac{m\omega}{\hbar}} \int_{-\infty}^{\infty} e^{-\xi^2}\left(\xi^4 - 3\xi^2\right) d\xi$$

$$= -2\alpha^2 \hbar^2 \sqrt{\frac{m\omega}{\hbar}}\, 2 \left(\sqrt{\pi} \cdot 12 \left(\frac{1}{2}\right)^5 - 3\sqrt{\pi}\, 2\, \frac{1}{8}\right)$$

for symmetric ex. function integral

$$= -2 \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \hbar^2 \sqrt{\frac{m\omega\pi}{\hbar}}\, 2 \left(\frac{12}{32} - \frac{6}{8}\right)$$

$$= -m\omega\hbar\, 2\left(\frac{24}{32} - \frac{12}{8}\right)$$

$$= -\left(-\frac{48}{32} + \frac{24}{32}\right) 2 m\omega\hbar$$

$$= +\frac{24}{32}\, m\omega\hbar\, 2$$

$$\frac{3 m\omega\hbar}{4}\, 2 = \frac{3 m\omega\hbar}{2}$$

$\langle x\rangle = \langle p\rangle = 0$ for both $n = 0$ and $n = 1 \ldots \infty$, because ψ^2 is an even function for the harmonic oscillator

b) $n = 0$

$$\sigma_x = \sqrt{\frac{\hbar}{2m\omega} - 0^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\frac{m\hbar\omega}{2}}$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar^2}{2^2}} = \frac{\hbar}{2} \checkmark$$

$n = 1$

$$\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\frac{3m\hbar\omega}{2}}$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar^2 3^2}{2^2}} = \frac{3}{2}\hbar \Rightarrow \frac{\hbar}{2} \checkmark$$

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c) $n = 0$

kinetic energy

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{m\hbar\omega}{2 \cdot 2m} = \frac{\hbar\omega}{4}$$

$n = 1$

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \frac{m\hbar\omega\, 3}{2 \cdot 2m} = \frac{3}{4}\hbar\omega$$

$n = 0$

$$\langle V\rangle = \frac{1}{2} m\omega^2 \langle x^2\rangle = \frac{\hbar\, m\omega^2}{4\, m\omega} = \frac{\hbar\omega}{4}$$

$n = 1$

$$\langle V\rangle = \frac{1}{2} m\omega^2 \langle x^2\rangle = \frac{3\hbar m\omega^2}{4 m\omega} = \frac{3}{4}\hbar\omega$$

$$\langle H\rangle = \langle T\rangle + \langle V\rangle = \begin{cases} \frac{1}{2}\hbar\omega = E_0 \checkmark & (n = 0) \\ \frac{3}{2}\hbar\omega = E_1 \checkmark & (n = 1) \end{cases}$$