Exercise 2.10

Thursday, 3 September 2020 16:11

$$\psi_2(x) = \frac{1}{\sqrt{2!}}(\hat{a}_+)^2 \psi_0$$

$$= \frac{1}{\sqrt{2}}\hat{a}_+\left(\frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega\hat{x}\right)\right)\psi_0$$

$$= \frac{1}{\sqrt{2}}\hat{a}_+\left(\frac{1}{\sqrt{2\hbar m\omega}}\left(+i^2\hbar\frac{\partial}{\partial x} + m\omega x\right)\right)\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2}}\hat{a}_+\left(\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(+\hbar\frac{m\omega 2x}{2\hbar}e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right)\right) \quad \psi_1$$

$$= \frac{1}{\sqrt{2}}\hat{a}_+\left(\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(m\omega x e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right)\right)$$

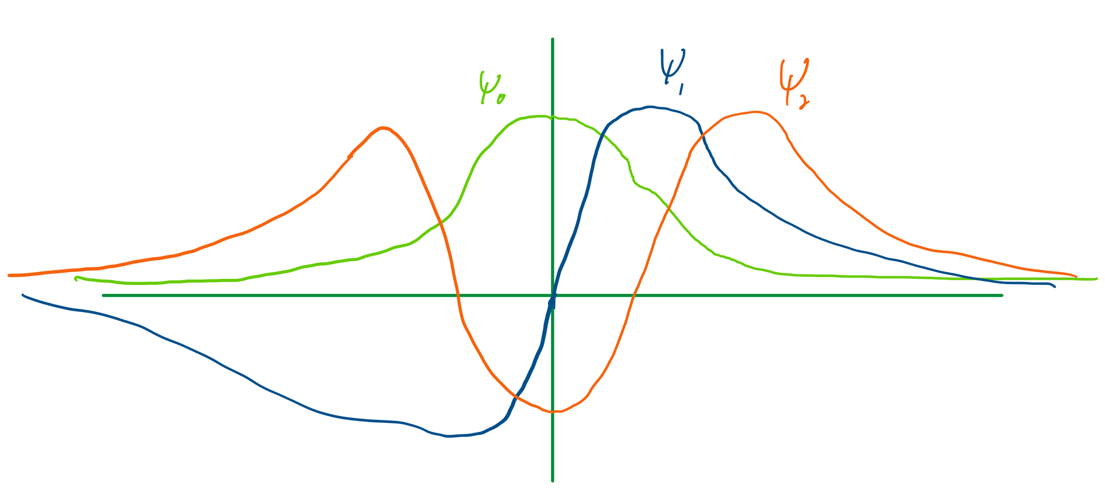
$$= \frac{1}{\sqrt{2}}\hat{a}_+\left(\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\sqrt{\frac{2m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}\right) \quad (2m\omega)$$

$$= \frac{1}{2\sqrt{\hbar m\omega}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right)\left(\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\sqrt{\frac{2m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \frac{1}{2\sqrt{\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\sqrt{\frac{2m\omega}{\hbar}}\left(-\hbar e^{-\frac{m\omega}{2\hbar}x^2} + \hbar 2x^2\frac{m\omega}{2\hbar}e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \frac{1}{\sqrt{2}\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar e^{-\frac{m\omega}{2\hbar}x^2} + 2m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}$$



ψ_0 and ψ_2 are even ; ψ_1 is odd

Thus $\int \psi_0^* \psi_1 \, dx = 0 = \int \psi_2^* \psi_1 \, dx$

Then, it remains to show that

$$\int \psi_2^* \psi_0 \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}e^{-\frac{m\omega}{2\hbar}x^2}\,dx$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}}\left(\int_{-\infty}^{\infty}\frac{2m\omega}{\hbar}x^2 e^{-\frac{m\omega x^2}{\hbar}}\,dx - \int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{\hbar}}\,dx\right)$$

even integrands, so use standard integral for $\int_0^\infty$ and multiply by 2

$$= \sqrt{\frac{m\omega}{2\pi\hbar}}\; 2\left(\frac{2m\omega}{\hbar}\sqrt{\pi}\,2\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3 - \sqrt{\pi}\,\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}}\left(\frac{\sqrt{\pi\hbar}}{2\sqrt{m\omega}} - \frac{\sqrt{\pi\hbar}}{2\sqrt{m\omega}}\right) = 0$$, so ψ_0 and ψ_2 are also orthogonal